

Traces of Intertwining Operators for the Yangian Double

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Abstract

The traces over infinite dimensional representations of the central extended Yangian double for the product of operators which intertwine these representations are calculated. For the special combinations of the intertwining operators the traces are identified with form factors of local operators in $SU(2)$ -invariant Thirring model. This identification is based on the identities which are deformed analogs of the Gauss-Manin connection identities for the hyperelliptic curves.

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1 Introduction

A big progress have been achieved last decades in the investigation of completely integrable quantum field theories. By the complete integrability we understand the possibility to write down explicit integral representation for an arbitrary form factor of any local operator in the model. One of the first methods which allows one to gain complete integrability was bootstrap program developed for Sin-Gordon and $SU(2)$ -invariant Thirring models in the works by F.A. Smirnov [S1]. Each time when realization of the bootstrap program led to success it always appeared as a miracle. The reason for such a miracle lies in the deep mathematical structure underlying the physical theories. One of such mathematical structures is a representation theory of infinite-dimensional Hopf algebras with the structure of doubles [D].

An application of such a symmetric ideology was realized recently in the series of papers by Kyoto group [DFJMN, JMMN, JM] devoted to investigation of quantum integrable lattice XXZ-type models in thermodynamic limit (for infinite lattices). The corresponding Hopf algebra for such models have been identified with quantum deformation of universal enveloping affine algebra $U_q(\widehat{\mathfrak{g}})$ with $|q| < 1$. Since a representation theory for this Hopf algebras is studied well enough, it is possible to calculate explicitly arbitrary physical quantity in the model. The main ingredient of this approach was a free field realization of infinite-dimensional integrable representations of $U_q(\widehat{\mathfrak{g}})$ [FJ].

The main goal of this paper is to expand this group-theoretical ideology to the continuous integrable models. In principle, there exists a possibility to obtain the answers in continuous theory using scaling limit from the corresponding lattice one [L]. But it is possible to do only for final objects, like integral formulas for form factors or correlation functions of the local operators while some intermediate ones (representation theory, for example) are lost in such a limit. We think that developing the direct group-theoretical methods in quantum integrable continuous field theories is an important problem.

Our main working example will be $SU(2)$ -invariant Thirring model, also known as $SU(2)$ chiral Gross-Neveu model or current-current perturbation of $SU(2)$ WZNW model at level 1. An infinite-dimensional Hopf algebra associated with this model is a Yangian double [B, BL, S2]. Our main observation is that in order to develop group-theoretical methods for complete integration of $SU(2)$ -invariant Thirring model one has to use a central extension of the Yangian double $DY(\widehat{\mathfrak{sl}_2})$, introduced recently in [K] and investigated in [KLP1]. For generalization to \mathfrak{sl}_n and \mathfrak{g}_n case see [I].

Organization of the paper is as follows. Section 2 is devoted to a short definition of central extension of the Yangian double $DY(\widehat{\mathfrak{sl}_2})$ and to realization of infinite dimensional representations of it in terms of free bosonic field. In section 3 we describe the properties of type I and type II intertwining operators for these representations. Next section is devoted to definition and calculation of traces for infinite-dimensional representation. In sections 5 and 6 particular cases of the general trace formula (4.11) are considered. It is shown that traces of the products of type I intertwining operators coincide with correlation functions of the inhomogeneous XXX model [N]. An analogous trace of type II operators is shown to be equal identically zero at “physical” value of shift parameter. In the last section the problem of identification of group-theoretical and bootstrap approaches to the calculation of form factors in completely integrable field theories is considered. This problem was not cleared up even in case of lattice models associated with $U_q(\widehat{\mathfrak{g}})$. We have found that this identification appears to be available because of special identities held between Smirnov’s integrals, which are deformed analogs of Gauss-Manin connection identities for the hyperelliptic integrals [M].

2 Central Extension of $DY(\widehat{\mathfrak{sl}_2})$

Central extension of the double of Yangian $DY(\mathfrak{sl}_2)$ is an infinite-dimensional Hopf algebra with central element c and generators $d, e_k, f_k, h_k, k \in \mathbb{Z}$, gathered into generating functions [K]

$$e(u) = \sum_{k \in \mathbb{Z}} e_k u^{-k-1}, \quad f(u) = \sum_{k \in \mathbb{Z}} f_k u^{-k-1}, \quad h^\pm(u) = 1 \pm \hbar \sum_{\substack{k \geq 0 \\ k < 0}} h_k u^{-k-1}, \quad (2.1)$$

with the following relations

$$[d, e(u)] = \frac{d}{du} e(u), \quad [d, f(u)] = \frac{d}{du} f(u), \quad [d, h^\pm(u)] = \frac{d}{du} h^\pm(u),$$

$$\begin{aligned}
e(u)e(v) &= \frac{u-v+\hbar}{u-v-\hbar} e(v)e(u) \\
f(u)f(v) &= \frac{u-v-\hbar}{u-v+\hbar} f(v)f(u) \\
h^\pm(u)e(v) &= \frac{u-v+\hbar}{u-v-\hbar} e(v)h^\pm(u) \\
h^+(u)f(v) &= \frac{u-v-\hbar-\hbar c}{u-v+\hbar-\hbar c} f(v)h^+(u) \\
h^-(u)f(v) &= \frac{u-v-\hbar}{u-v+\hbar} f(v)h^-(u) \\
h^+(u)h^-(v) &= \frac{u-v+\hbar}{u-v-\hbar} \cdot \frac{u-v-\hbar-\hbar c}{u-v+\hbar-\hbar c} h^-(v)h^+(u) \\
[e(u), f(v)] &= \frac{1}{\hbar} (\delta(u-(v+\hbar c))h^+(u) - \delta(u-v)h^-(v))
\end{aligned} \tag{2.2}$$

where

$$\delta(u-v) = \sum_{n+m=-1} u^n v^m, \quad \delta(u-v)g(u) = \delta(u-v)g(v).$$

For the comultiplication formulas in $DY(\widehat{\mathfrak{sl}_2})$ see [K], [KLP1].

Let \mathcal{H} be a Heisenberg algebra generated by free bosons with zero modes $a_{\pm n}$, $n = 1, 2, \dots$, a_0 , p with commutation relations

$$[a_n, a_m] = n\delta_{n+m,0}, \quad [p, a_0] = 2.$$

We use the following generating functions of elements from \mathcal{H} :

$$a_+(z) = \sum_{n \geq 1} \frac{a_n}{n} z^{-n} - p \log z, \quad a_-(z) = \sum_{n \geq 1} \frac{a_{-n}}{n} z^n + \frac{a_0}{2}, \tag{2.3}$$

$$a(z) = a_+(z) - a_-(z), \quad \phi_\pm(z) = \exp a_\pm(z), \tag{2.4}$$

$$[a_+(z), a_-(y)] = -\log(z-y), \quad |y| < |z|. \tag{2.5}$$

Let $V_i, i = 0, 1$ be formal power extensions of the Fock spaces

$$V_i = \mathbb{C}[[a_{-1}, \dots, a_{-n}, \dots]] \otimes (\oplus_{n \in \mathbb{Z}+i/2} \mathbb{C}e^{na_0}) \tag{2.6}$$

with the action of bosons on these spaces

$$\begin{aligned}
a_n &= \text{the left multiplication by } a_n \otimes 1 \text{ for } n < 0, \\
&= [a_n, \cdot] \otimes 1 \text{ for } n > 0, \\
e^{n_1 a_0}(a_{-j_k} \cdots a_{-j_1} \otimes e^{n_2 a_0}) &= a_{-j_k} \cdots a_{-j_1} \otimes e^{(n_1+n_2)a_0}, \\
u^p(a_{-j_k} \cdots a_{-j_1} \otimes e^{n a_0}) &= u^{2n} a_{-j_k} \cdots a_{-j_1} \otimes e^{n a_0}.
\end{aligned}$$

Proposition 1. *The following $\text{End } V_i$ -valued functions satisfy commutation relations (2.2) with $c = 1$:*

$$\begin{aligned}
e(u) &= \mu_-(u-\hbar)\phi_+^{-1}(u), \quad f(u) = \mu_-^{-1}(u)\phi_+(u), \\
h^+(u) &= \phi_+(u-\hbar)\phi_+^{-1}(u), \quad h^-(u) = \phi_-(u-\hbar)\phi_-^{-1}(u+\hbar), \\
e^{\gamma d}\phi_\pm(u) &= \phi_\pm(u+\gamma)e^{\gamma d}, \quad e^{\gamma d}(1 \otimes 1) = 1 \otimes 1,
\end{aligned} \tag{2.7}$$

where

$$\mu_-(u) = \phi_-(u+\hbar)\phi_-(u). \tag{2.8}$$

Due to Proposition 1, the Fock spaces $V_i, i = 0, 1$ are $DY(\widehat{\mathfrak{sl}_2})$ -modules. We call them basic (level 1) representations of $DY(\widehat{\mathfrak{sl}_2})$.

3 Intertwining Operators for Basic Representations

Let $\eta_+(z)$ be the following End V_i -valued function (field):

$$\eta_+(z) = \lim_{K \rightarrow \infty} (2\hbar K)^{-p/2} \prod_{k=0}^K \frac{\phi_+(z - 2k\hbar)}{\phi_+(z - \hbar - 2k\hbar)} \quad (3.1)$$

The field $\eta_+(z)$ is well defined and satisfies a property

$$\begin{aligned} \eta_+(z)\eta_+(z - \hbar) &= \lim_{K \rightarrow \infty} (2\hbar K)^{-p} \prod_{k=0}^K \frac{\phi_+(z - 2k\hbar)}{\phi_+(z - 2\hbar - 2k\hbar)} \\ &= \phi_+(z) \lim_{K \rightarrow \infty} (2\hbar K)^{-p} \phi_+^{-1}(z - 2\hbar - 2K\hbar) \\ &= \phi_+(z)(-1)^p \end{aligned} \quad (3.2)$$

where the operator $(-1)^p$ acts on the module V_i multiplying it by $(-1)^i$. We can rewrite $\eta_+(z)$ in an equivalent form using the Stirling formula

$$\begin{aligned} \eta_+(z) &= (2\hbar)^{-p/2} \left(\frac{\Gamma(\frac{1}{2} - \frac{z}{2\hbar})}{\Gamma(-\frac{z}{2\hbar})} \right)^{-p} \\ &\times \lim_{K \rightarrow \infty} \prod_{k=0}^K \exp \left(\sum_{n=1}^{\infty} \frac{a_n}{n} [(z - 2k\hbar)^{-n} - (z - \hbar - 2k\hbar)^{-n}] \right) \end{aligned} \quad (3.3)$$

This presentation is useful to investigate the classical limit of the field $\eta_+(z)$

$$\lim_{\hbar \rightarrow 0} \eta_+(z) = \phi_+^{1/2}(z)(-1)^{p/2} \quad (3.4)$$

which shows that the formula (3.1) can be treated as a deformed square root.

Let us define the intertwining operators

$$\Phi^{(i)}(\alpha) : V_i \rightarrow V_{1-i} \otimes V_\alpha, \quad \Phi^{*(i)}(\alpha) : V_i \otimes V_\alpha \rightarrow V_{1-i}, \quad (3.5)$$

$$\Psi^{(i)}(\beta) : V_i \rightarrow V_\beta \otimes V_{1-i}, \quad \Psi^{*(i)}(\beta) : V_\beta \otimes V_i \rightarrow V_{1-i} \quad (3.6)$$

which commute with the action of the Yangian double. Here V_z in (3.5) and (3.6) denotes two-dimensional evaluation module

$$V_z = V \otimes \mathbb{C}[[z, z^{-1}]], \quad V = \mathbb{C}v_+ \otimes \mathbb{C}v_-$$

dependent on the spectral parameter z [KT]. The components of the intertwining operators are defined as follows

$$\begin{aligned} \Phi^{(i)}(\alpha)v &= \Phi_+^{(i)}(\alpha)v \otimes v_+ + \Phi_-^{(i)}(\alpha)v \otimes v_- , \quad \Phi^{*(i)}(\alpha)(v \otimes v_\pm) = \Phi_\pm^{*(i)}(\alpha)v , \\ \Psi^{(i)}(\beta)v &= v_+ \otimes \Psi_+^{(i)}(\beta)v + v_- \otimes \Psi_-^{(i)}(\beta)v , \quad \Psi^{*(i)}(\beta)(v_\pm \otimes v) = \Psi_\pm^{*(i)}(\beta)v , \end{aligned}$$

where $v \in V_i$. Fix also the normalization of these operators

$$\begin{aligned} \Phi^{(i)}(\alpha)u_i &= (-\alpha)^{-i/2} \varepsilon_{1-i} u_{1-i} \otimes v_{\varepsilon_i} + \dots, \\ \Psi^{*(i)}(\beta)(v_{\varepsilon_{1-i}} \otimes u_i) &= (-\beta)^{-i/2} u_{1-i} + \dots, \quad \varepsilon_0 = -, \quad \varepsilon_1 = +, \end{aligned} \quad (3.7)$$

where we denote

$$u_0 = 1 \otimes 1 \quad \text{and} \quad u_1 = 1 \otimes e^{a_0/2}.$$

Normalization condition (3.7) allows us to write down precise expressions for

$$\Phi_\varepsilon = \Phi_\varepsilon^{(0)} \oplus \Phi_\varepsilon^{(1)} : V_0 \oplus V_1 \rightarrow V_1 \oplus V_0 \quad \text{and} \quad \Psi_\varepsilon^* = \Psi_\varepsilon^{*(0)} \oplus \Psi_\varepsilon^{*(1)} : V_0 \oplus V_1 \rightarrow V_1 \oplus V_0$$

without dependence on the index i . We have the following [KLP1]

Proposition 2. *Intertwining operators (3.5) have the free field realization:*

$$\Phi_-(\alpha) = \phi_-(\alpha + \hbar)\eta_+^{-1}(\alpha), \quad (3.8)$$

$$\Phi_+(\alpha) = \Phi_-(\alpha)f_0 - f_0\Phi_-(\alpha), \quad (3.9)$$

$$\Phi_\nu^{*(i)}(\alpha) = \nu(-1)^i \Phi_{-\nu}^{(i)}(\alpha - \hbar), \quad \nu = \pm, \quad (3.10)$$

$$\Psi_-^*(\beta) = \phi_-^{-1}(\beta)\eta_+(\beta), \quad (3.11)$$

$$\Psi_+^*(\beta) = e_0\Psi_-^*(\beta) - \Psi_-^*(\beta)e_0, \quad (3.12)$$

$$\Psi_\varepsilon^{(i)}(\beta) = \varepsilon(-1)^{-i}\Psi_{-\varepsilon}^{*(i)}(\beta - \hbar), \quad \varepsilon = \pm. \quad (3.13)$$

These intertwining operators satisfy commutation relations

$$\Phi_{\varepsilon_2}(\alpha_2)\Phi_{\varepsilon_1}(\alpha_1) = R_{\varepsilon_1\varepsilon_2}^{\varepsilon'_1\varepsilon'_2}(\alpha_1 - \alpha_2)\Phi_{\varepsilon'_1}(\alpha_1)\Phi_{\varepsilon'_2}(\alpha_2), \quad (3.14)$$

$$\Psi_{\varepsilon_1}^*(\beta_1)\Psi_{\varepsilon_2}^*(\beta_2) = -R_{\varepsilon_1\varepsilon_2}^{\varepsilon'_1\varepsilon'_2}(\beta_1 - \beta_2)\Psi_{\varepsilon'_2}^*(\beta_2)\Psi_{\varepsilon'_1}^*(\beta_1), \quad (3.15)$$

$$\Phi_{\varepsilon_1}(\alpha)\Psi_{\varepsilon_2}^*(\beta) = \tau(\alpha - \beta)\Psi_{\varepsilon_2}^*(\beta)\Phi_{\varepsilon_1}(\alpha), \quad (3.16)$$

$$g \sum_{\varepsilon} \Phi_{\varepsilon}^{*(1-i)}(\alpha)\Phi_{\varepsilon}^{(i)}(\alpha) = -\text{id}, \quad (3.17)$$

$$g\Phi_{\varepsilon_1}^{(1-i)}(\alpha)\Phi_{\varepsilon_2}^{*(i)}(\alpha) = \delta_{\varepsilon_1\varepsilon_2} \text{id}, \quad (3.18)$$

$$g^{-1}\Psi_{\varepsilon_1}^{(1-i)}(\beta_1)\Psi_{\varepsilon_2}^{*(i)}(\beta_2) = \frac{\delta_{\varepsilon_1\varepsilon_2}}{\beta_1 - \beta_2} + o(\beta_1 - \beta_2), \quad (3.19)$$

where R -matrix is given by

$$R(z) = r(z)\overline{R}(z) \quad (3.20)$$

and

$$r(z) = \frac{\Gamma(\frac{1}{2} - \frac{z}{2\hbar})\Gamma(1 + \frac{z}{2\hbar})}{\Gamma(\frac{1}{2} + \frac{z}{2\hbar})\Gamma(1 - \frac{z}{2\hbar})}, \quad (3.21)$$

$$\tau(z) = \frac{\Gamma(1 + \frac{z}{2\hbar})\Gamma(-\frac{z}{2\hbar})}{\Gamma(\frac{1}{2} + \frac{z}{2\hbar})\Gamma(\frac{1}{2} - \frac{z}{2\hbar})} = -\text{ctg} \frac{\pi z}{2\hbar},$$

$$\overline{R}(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(z) & c(z) & 0 \\ 0 & c(z) & b(z) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (3.22)$$

$$b(z) = \frac{z}{z + \hbar}, \quad c(z) = \frac{\hbar}{z + \hbar}, \quad g = \sqrt{\frac{2\hbar}{\pi}}.$$

One can easily check that R -matrix (3.20) satisfies the unitary and the crossing symmetry conditions

$$R(z)R(-z) = 1,$$

$$(C \otimes \text{id})R(z)(C \otimes \text{id}) = R^{t_1}(-z - \hbar)$$

with the charge conjugation matrix

$$C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (3.23)$$

and $R^{t_1}(z)$ means the transposition with respect to first space.

The verification of these commutation relations is based on the following normal ordering relations (see details in [KLP1])

$$\Phi_{-}(\alpha_2)\Phi_{-}(\alpha_1) = (2\hbar)^{1/2} \frac{\Gamma\left(1 + \frac{\alpha_1 - \alpha_2}{2\hbar}\right)}{\Gamma\left(\frac{1}{2} + \frac{\alpha_1 - \alpha_2}{2\hbar}\right)} : \Phi_{-}(\alpha_2)\Phi_{-}(\alpha_1) : , \quad (3.24)$$

$$\Psi_{-}^{*}(\beta_1)\Psi_{-}^{*}(\beta_2) = (2\hbar)^{1/2} \frac{\Gamma\left(\frac{1}{2} - \frac{\beta_1 - \beta_2}{2\hbar}\right)}{\Gamma\left(-\frac{\beta_1 - \beta_2}{2\hbar}\right)} : \Psi_{-}^{*}(\beta_1)\Psi_{-}^{*}(\beta_2) : , \quad (3.25)$$

$$\Psi_{\varepsilon}^{*}(\beta)\Phi_{\nu}(\alpha) = (2\hbar)^{-1/2} \frac{\Gamma\left(\frac{1}{2} + \frac{\alpha - \beta}{2\hbar}\right)}{\Gamma\left(1 + \frac{\alpha - \beta}{2\hbar}\right)} : \Psi_{\varepsilon}^{*}(\beta)\Phi_{-}(\alpha) : , \quad (3.26)$$

$$\begin{aligned} \Phi_{-}(\alpha)f(u) &= \frac{1}{u - \alpha} : \Phi_{-}(\alpha)f(u) : \\ f(u)\Phi_{-}(\alpha) &= \frac{1}{u - \alpha - \hbar} : \Phi_{-}(\alpha)f(u) : \end{aligned} \quad (3.27)$$

$$\begin{aligned} \Psi_{-}^{*}(\beta)e(v) &= \frac{1}{v - \beta - \hbar} : \Psi_{-}^{*}(\beta)e(v) : \\ e(v)\Psi_{-}^{*}(\beta) &= \frac{1}{v - \beta} : \Psi_{-}^{*}(\beta)e(v) : \end{aligned} \quad (3.28)$$

$$\begin{aligned} e(v)\Phi_{-}(\alpha) &= \Phi_{-}(\alpha)e(v) = (v - \alpha - \hbar) : \Phi_{-}(\alpha)e(v) : \\ f(u)\Psi_{-}^{*}(\beta) &= \Psi_{-}^{*}(\beta)f(u) = (u - \beta) : \Psi_{-}^{*}(\beta)f(u) : \end{aligned}$$

These normal ordering formulas are used in particular to represent the second components of the intertwining operators where some contour integrations over variables u and v are supposed. Let us specify the contours in these integrals. By inspection of (3.27) we can easily find that the point $\alpha + \hbar$ should be inside the contour C in the definition of the operator $\Phi_{+}(\alpha)$

$$\Phi_{+}(\alpha) = \Phi_{-}(\alpha)f_0 - f_0\Phi_{-}(\alpha) = \int_C \frac{du}{2i\pi} (\Phi_{-}(\alpha)f(u) - f(u)\Phi_{-}(\alpha)) , \quad (3.29)$$

while the point α should be outside C . For the type II intertwining operators the picture is inverse. The contour \tilde{C} in definition of the operator $\Psi_{+}^{*}(\beta)$

$$\Psi_{+}^{*}(\beta) = e_0\Psi_{-}^{*}(\beta) - \Psi_{-}^{*}(\beta)e_0 = \int_{\tilde{C}} \frac{dv}{2i\pi} (e(v)\Psi_{-}^{*}(\beta) - \Psi_{-}^{*}(\beta)e(v)) \quad (3.30)$$

should include the point β but exclude the point $\beta + \hbar$. The form of the contours C and \tilde{C} depends on the presence of other operators and should be specified in each particular case of the product of intertwining operators.

4 Trace Formula

The goal of this section is to obtain the trace of the product of arbitrary number of type I and type II intertwining operators. We need the following

Proposition 2. *Let*

$$O = \prod_j \phi_{-}^{k_j}(w_j) \prod_k \eta_{+}^{p_k}(z_k) \quad (4.1)$$

be an operator acting in the infinite dimensional Fock space V_0 . Under the restrictions for the integers k_j and p_k

$$\sum_j k_j = 0 \quad \sum_k p_k = 0 \quad (4.2)$$

a ratio

$$\frac{\text{tr}(e^{\gamma d} O)}{\text{tr}(e^{\gamma d})}$$

of traces over this infinite dimensional representation space can be calculated using the formula

$$\frac{\text{tr}(e^{\gamma d} O)}{\text{tr}(e^{\gamma d})} = \prod_{m=1}^{\infty} \exp \left(\text{res}_{x,y} \frac{<a_+(x-m\gamma)[O-1]a'_-(y)>}{x-y} \right) \quad (4.3)$$

and is equal to

$$\frac{1}{\text{tr}(e^{\gamma d})} \text{tr} \left(e^{\gamma d} \prod_j \phi_-^{k_j}(w_j) \prod_k \eta_+^{p_k}(z_k) \right) = \prod_{m=1}^{\infty} \prod_{k=0}^{\infty} \prod_{j,k} \frac{(z_k - w_j - \hbar - m\gamma - 2k\hbar)^{k_j p_k}}{(z_k - w_j - m\gamma - 2k\hbar)^{k_j p_k}}, \quad (4.4)$$

where an integral representation for the r.h.s. of (4.4) is ($\text{Re } \gamma/\hbar > 0$)

$$\exp \left(\frac{1}{4} \int_0^{\infty} \frac{dx}{x} \sum_{j,k} k_j p_k \frac{e^{\frac{(\hbar-\gamma)x}{2\hbar}} e^{\frac{(z_k-w_j)x}{\hbar}}}{\text{sh} \frac{\gamma x}{2\hbar} \text{ch} \frac{x}{2}} \right). \quad (4.5)$$

Formula (4.3) is an analog of the formula (8.6) given in the book [JM] for $U_q(\widehat{\mathfrak{sl}}_2)$. To obtain (4.3) one has to pass from summation over powers of different modes of free bosonic field, to the summation over products of the free field taken in different points. In the basis constructed from these fields an operator $e^{\gamma d}$ acts by shift of the spectral parameter. The proof of (4.3) is complicated but straightforward and can be found in [C]. A significant difference of trace calculations in Yangian case in contrast to quantum affine one [JM] is that only ratio of traces is well defined in a sense that corresponding infinite products are convergent.

Because of the property (3.2) of the field $\eta_+(z)$ and definition (2.8) of the field $\eta_-(z)$ the formula (4.4) is sufficient to calculate (4.3) when the operator O is a product of different combinations of free fields $\phi_-(z)$, $\phi_+(z)$, $\mu_-(z)$ and $\eta_+(z)$.

Let us apply above formulas for computation of the trace over the space V_0 of the following product of type I and type II intertwining operators

$$\Psi_{\varepsilon_N}^*(\beta_N) \dots \Psi_{\varepsilon_1}^*(\beta_1) \Phi_{\nu_M}(\alpha_M) \dots \Phi_{\nu_1}(\alpha_1) \quad (4.6)$$

with the numbers of “plus” components of the operators

$$\{\#j | \varepsilon_j = +\} = n \quad \{\#i | \nu_i = +\} = m.$$

First, we have to normal order the product (4.6) using formulas of the previous section. We can rewrite 4.6 ($\alpha_{ij} = \alpha_i - \alpha_j$ and $\beta_{ij} = \beta_i - \beta_j$) as following:

$$\begin{aligned} & \prod_{j=1}^N \Psi_{\varepsilon_j}^*(\beta_j) \prod_{i=1}^M \Phi_{\nu_i}(\alpha_i) = (-\hbar)^{n+m} (2\hbar)^{\frac{N(N-1)+M(M-1)-2NM}{4}} \\ & \times \prod_{i < j} \frac{\Gamma\left(\frac{1}{2} + \frac{\beta_{ij}}{2\hbar}\right)}{\Gamma\left(\frac{\beta_{ij}}{2\hbar}\right)} \prod_{i < j} \frac{\Gamma\left(1 + \frac{\alpha_{ij}}{2\hbar}\right)}{\Gamma\left(\frac{1}{2} + \frac{\alpha_{ij}}{2\hbar}\right)} \prod_{j=1}^N \prod_{i=1}^M \frac{\Gamma\left(\frac{1}{2} + \frac{\alpha_i - \beta_j}{2\hbar}\right)}{\Gamma\left(1 + \frac{\alpha_i - \beta_j}{2\hbar}\right)} \\ & \times \int_{\tilde{C}} \dots \int \frac{dv_k}{2\pi i} \frac{\tilde{P}_{\varepsilon}(v; \beta) \prod_{i < j} (v_i - v_j)(v_i - v_j - \hbar)}{\prod_{i=1}^n \prod_{j=1}^M (v_i - \beta_j)(v_i - \beta_j - \hbar)} \\ & \times \int_C \dots \int \frac{du_p}{2\pi i} \frac{P_{\nu}(u; \alpha) \prod_{i < j} (u_i - u_j)(u_i - u_j + \hbar)}{\prod_{i=1}^m \prod_{j=1}^M (u_i - \alpha_j)(u_i - \alpha_j - \hbar)} \\ & \times \frac{\prod_{k=1}^n \prod_{i=1}^M (v_k - \alpha_p - \hbar) \prod_{p=1}^m \prod_{j=1}^N (u_p - \beta_j)}{\prod_{k=1}^n \prod_{p=1}^m (u_p - v_k)(u_p - v_k + \hbar)} O_- O_+, \end{aligned}$$

where

$$O_- = \prod_{j=1}^N \phi_-^{-1}(\beta_j) \prod_{i=1}^M \phi_-(\alpha_i + \hbar) \prod_{k=1}^n \mu_-(v_k - \hbar) \prod_{p=1}^m \mu_-^{-1}(u_p) \quad (4.7)$$

$$O_+ = \prod_{j=1}^N \eta_+(\beta_j) \prod_{i=1}^M \eta_+^{-1}(\alpha_i) \prod_{k=1}^n \phi_+^{-1}(v_k) \prod_{p=1}^m \phi_+(u_p) \quad (4.8)$$

and for

$$\{b_1 < \dots < b_n\} = \{j \mid \varepsilon_j = +\} \quad \text{and} \quad \{a_1 < \dots < a_n\} = \{i \mid \nu_i = +\} \quad (4.9)$$

we define the polynomials

$$P_\nu(u; \alpha) = \prod_{p=1} \prod_{i < a_p} (u_p - \alpha_i) \prod_{i > a_p} (u_p - \alpha_i - \hbar) ,$$

$$\tilde{P}_\varepsilon(v; \beta) = \prod_{k=1} \prod_{j > b_k} (v_k - \beta_j) \prod_{j < b_k} (v_k - \beta_j - \hbar) .$$

Applying formula (4.4) to the operators $O_- O_+$ given by (4.7) and (4.8) we see that the condition (4.2) for the existence of this trace can be written as

$$\frac{N - M}{2} = n - m \quad (4.10)$$

and the trace of the product (4.6) is equal to

$$\begin{aligned} & \frac{1}{\text{tr}(e^{\gamma d})} \text{tr} \left(e^{\gamma d} \prod_{j=1}^N \Psi_{\varepsilon_j}^*(\beta_j) \prod_{i=1}^M \Phi_{\nu_i}(\alpha_i) \right) \\ &= A_{NM}^{nm} \prod_{j < j'} \tilde{\zeta}(\beta_j - \beta_{j'}) \prod_{i < i'} \zeta(\alpha_i - \alpha_{i'}) \prod_{i,j} \bar{\zeta}(\alpha_i - \beta_j) \\ & \times \prod_{k=1}^n \int_{\tilde{C}} \frac{dv_k}{2\pi i} \prod_{j=1}^N \Gamma\left(\frac{\beta_j - v_i}{\gamma}\right) \Gamma\left(\frac{v_i - \beta_j - \hbar}{\gamma}\right) \frac{\tilde{P}_\varepsilon(v; \beta)}{\gamma^{n(N-1)}} \\ & \times \prod_{k < k'} \frac{\sin \pi \frac{v_k - v_{k'}}{\gamma}}{\Gamma\left(\frac{v_k - v_{k'} - \hbar}{\gamma}\right) \Gamma\left(\frac{v_{k'} - v_k - \hbar + \gamma}{\gamma}\right)} \\ & \times \prod_{p=1}^m \int_C \frac{du_p}{2\pi i} \prod_{i=1}^M \Gamma\left(\frac{u_p - \alpha_i}{\gamma}\right) \Gamma\left(\frac{\alpha_i - u_p + \hbar}{\gamma}\right) \frac{P_\nu(u; \alpha)}{\gamma^{m(M-1)}} \\ & \times \prod_{p < p'} \frac{\sin \pi \frac{u_p - u_{p'}}{\gamma}}{\Gamma\left(\frac{u_p - u_{p'} + \hbar}{\gamma}\right) \Gamma\left(\frac{u_{p'} - u_p + \hbar + \gamma}{\gamma}\right)} \\ & \times \frac{\prod_{p=1}^m \prod_{j=1}^N \sin \pi \frac{u_p - \beta_j}{\gamma} \prod_{k=1}^n \prod_{i=1}^M \sin \pi \frac{v_k - \alpha_i}{\gamma}}{\prod_{p=1}^m \prod_{k=1}^n \sin \pi \frac{u_p - v_k}{\gamma} \sin \pi \frac{u_p - v_k + \hbar}{\gamma}} \end{aligned} \quad (4.11)$$

This formula in particular case $\gamma = 2\hbar$ has been written in [L] and can be obtained by means of the scaling limit from general trace formula given in [JM] for the quantum affine algebra.

In (4.11) the numerical constant A_{NM}^{nm} is

$$A_{NM}^{nm} = \left(\frac{2\hbar}{\sqrt{\pi}\gamma} \right)^{(N-M)^2/4} \frac{(-1)^{Nn+Mm} (-\hbar)^{n+m} \pi^{nm+n/2+m/2} \tilde{G}^{N/4}(0) G^{M/4}(0)}{(2\hbar)^{(N+M)/4} \gamma^{2n+2m} \Gamma\left(1 - \frac{\hbar}{\gamma}\right)^{(N^2+4n)/4} \Gamma\left(1 + \frac{\hbar}{\gamma}\right)^{(M^2+4m)/4}} \quad (4.12)$$

and we introduced functions

$$\begin{aligned}\zeta(\alpha) &= \left[\frac{\Gamma\left(1 + \frac{\alpha}{2\hbar}\right)}{\Gamma\left(\frac{1}{2} + \frac{\alpha}{2\hbar}\right)} \frac{G(\alpha)}{G^{1/2}(0)} \right], \\ \tilde{\zeta}(\beta) &= \left[\frac{\Gamma\left(\frac{1}{2} + \frac{\beta}{2\hbar}\right)}{\Gamma\left(\frac{\beta}{2\hbar}\right)} \frac{\tilde{G}(\beta)}{\tilde{G}^{1/2}(0)} \right], \\ \bar{\zeta}(z) &= \left[\frac{\Gamma\left(\frac{1}{2} + \frac{z}{2\hbar}\right)}{\Gamma\left(1 + \frac{z}{2\hbar}\right)} \bar{G}(z) \right]\end{aligned}$$

and

$$\begin{aligned}G(\alpha) &= \prod_{m=1}^{\infty} \prod_{k=0}^{\infty} \frac{(-\hbar - 2k\hbar - m\gamma)(-3\hbar - 2k\hbar - m\gamma)}{(-2k\hbar - m\gamma)(-2\hbar - 2k\hbar - m\gamma)} \\ &\quad \times \frac{(\alpha - \hbar - 2k\hbar - m\gamma)(-\alpha - \hbar - 2k\hbar - m\gamma)}{(\alpha - 2\hbar - 2k\hbar - m\gamma)(-\alpha - 2\hbar - 2k\hbar - m\gamma)},\end{aligned}\quad (4.13)$$

$$\begin{aligned}\tilde{G}(\beta) &= \prod_{m=1}^{\infty} \prod_{k=0}^{\infty} \frac{(-2k\hbar - m\gamma)(-2\hbar - 2k\hbar - m\gamma)}{(-\hbar - 2k\hbar - m\gamma)(\hbar - 2k\hbar - m\gamma)} \\ &\quad \times \frac{(\beta - 2k\hbar - m\gamma)(-\beta - 2k\hbar - m\gamma)}{(\beta - \hbar - 2k\hbar - m\gamma)(-\beta - \hbar - 2k\hbar - m\gamma)},\end{aligned}\quad (4.14)$$

$$\begin{aligned}\bar{G}(z) &= \prod_{m=1}^{\infty} \prod_{k=0}^{\infty} \frac{(-2k\hbar - m\gamma)}{(-2\hbar - 2k\hbar - m\gamma)} \\ &\quad \times \frac{(z - \hbar - 2k\hbar - m\gamma)(-z - 2\hbar - 2k\hbar - m\gamma)}{(z - 2k\hbar - m\gamma)(-z - \hbar - 2k\hbar - m\gamma)}.\end{aligned}\quad (4.15)$$

Functions (4.13), (4.14) and (4.15) are well defined because the infinite product of rational functions [Ba]

$$\prod_{k_1, \dots, k_n \geq 0} \frac{\prod_m (a_m + \sum k_j \omega_j)}{\prod_p (b_p + \sum k_j \omega_j)} \quad (4.16)$$

is well defined as function of the variables a_m and b_p if the following constraints for these variables are satisfied

$$\sum_m (a_m)^q = \sum_p (b_p)^q, \quad q = 0, 1, \dots, n. \quad (4.17)$$

In particular, for $n = 1$ the function (4.16) is equal to the ratio of Γ -functions. The simplest way to prove (4.17) is to use an integral representation for the function (4.16).

The functions $G(\alpha)$, $\tilde{G}(\beta)$ and $\bar{G}(z)$ have the following integral representations in appropriate regions of parameters α, β, z ($\text{Re } \gamma/\hbar > 0$)

$$G(\alpha) = \exp \left(\int_0^\infty \frac{dx}{x} \frac{\text{th} \frac{x}{2} \text{sh} \frac{(\alpha + \hbar)x}{2\hbar} \text{sh} \frac{(\alpha - \hbar)x}{2\hbar}}{\text{sh} \frac{\gamma x}{2\hbar}} \exp \left(-\frac{(\gamma - 2\hbar)x}{2\hbar} \right) \right), \quad (4.18)$$

$$\tilde{G}(\beta) = \exp \left(\int_0^\infty \frac{dx}{x} \frac{\text{th} \frac{x}{2} \text{sh} \frac{(\beta + \hbar)x}{2\hbar} \text{sh} \frac{(\beta - \hbar)x}{2\hbar}}{\text{sh} \frac{\gamma x}{2\hbar}} \exp \left(-\frac{\gamma x}{2\hbar} \right) \right), \quad (4.19)$$

$$\bar{G}(z) = \exp \left(- \int_0^\infty \frac{dx}{x} \frac{\text{th} \frac{x}{2} \text{sh} \frac{(z + \hbar)x}{2\hbar} \text{sh} \frac{zx}{2\hbar}}{\text{sh} \frac{\gamma x}{2\hbar}} \exp \left(\frac{(\gamma - \hbar)x}{2\hbar} \right) \right) \quad (4.20)$$

and they satisfy the properties

$$G(z) = G(-z), \quad G(\hbar) = 1, \quad \tilde{G}(z) = \tilde{G}(-z), \quad \tilde{G}(\hbar) = 1, \quad \overline{G}(0) = 1, \quad (4.21)$$

$$G(z)\tilde{G}(z) = \frac{z \sin(\pi\hbar/\gamma)}{\hbar \sin(\pi z/\gamma)}, \quad \overline{G}(z)\overline{G}(-z) = \overline{G}(z)\overline{G}(z - \hbar) = \frac{\pi z}{\gamma} \frac{1}{\sin(\pi z/\gamma)}. \quad (4.22)$$

Properties (4.21) are useful for checking accordance of the formula (4.11) with the normalization of the intertwining operators (3.17)–(3.19).

The last thing which we should to do is to specify the contours C and \tilde{C} in (4.11). The correct choice of the contours is dictated by (3.29), (3.30) and prescriptions how to use the formula (4.4). The contour C for the integration over u_p , $p = 1, \dots, m$ is such that ($r = 0, 1, 2, \dots$)

$$\begin{aligned} \alpha_j + \hbar + r\gamma & \text{ are inside } C, \quad \alpha_j - r\gamma & \text{ are outside } C, \quad j = 1, \dots, M \\ v_k - \hbar + \gamma + r\gamma, \quad v_k + \gamma + r\gamma & \text{ are inside } C, \\ v_k - \hbar - r\gamma, \quad v_k - r\gamma & \text{ are outside } C, \quad k = 1, \dots, n \end{aligned} \quad (4.23)$$

and the contour \tilde{C} for the integration over v_k , $k = 1, \dots, n$ is such that ($r = 0, 1, 2, \dots$)

$$\begin{aligned} \beta_i + r\gamma & \text{ are inside } \tilde{C}, \quad \beta_i + \hbar - r\gamma & \text{ are outside } \tilde{C}, \quad i = 1, \dots, N \\ u_p + \hbar + r\gamma, \quad u_p + r\gamma & \text{ are inside } \tilde{C}, \\ u_p + \hbar - \gamma - r\gamma, \quad u_p - \gamma - r\gamma & \text{ are outside } \tilde{C}, \quad p = 1, \dots, m. \end{aligned} \quad (4.24)$$

Also we should remark that both contours cross the infinity point along the line $z = i\hbar t$, $t \in \mathbb{R}$ in the complex planes corresponding to the integration variables v_k and u_p because they are pinched from both sides of this line by the poles of the Γ -functions. If the integrand in the general formula (4.11) has a pole in the point ∞ then we should understand corresponding integrals as principal value integrals (see next section).

It is quite difficult to work with general formula (4.11) so in next four sections we will consider particular cases of this formula which are of special interest for us.

5 Correlation Functions

The experience of group-theoretical approach in quantum integrable lattice models [JM] teaches that some combinations of type I intertwining operators can be identified with generating functions of the local operators in the corresponding quantum integrable model. Therefore the calculation of traces of type I intertwining operators gives in particular the correlation functions of local operators. In this case we have from (4.11) for $N = n = 0$ and $M = 2m$ due to (4.10)

$$\begin{aligned} \frac{1}{\text{tr}(e^{\gamma d})} \text{tr} \left(e^{\gamma d} \prod_{i=1}^{2m} \Phi_{\nu_i}(\alpha_i) \right) &= A_{0,2m}^{0,m} \prod_{i < i'} \zeta(\alpha_i - \alpha_{i'}) \\ &\times \prod_{p=1}^m \int_C \frac{du_p}{2\pi i} \prod_{i=1}^{2m} \Gamma \left(\frac{u_p - \alpha_i}{\gamma} \right) \Gamma \left(\frac{\alpha_i - u_p + \hbar}{\gamma} \right) \frac{P_\nu(u; \alpha)}{\gamma^{m(2m-1)}} \\ &\times \prod_{p < p'} \frac{\sin \pi \frac{u_p - u_{p'}}{\gamma}}{\Gamma \left(\frac{u_p - u_{p'} + \hbar}{\gamma} \right) \Gamma \left(\frac{u_{p'} - u_p + \hbar + \gamma}{\gamma} \right)} \end{aligned} \quad (5.1)$$

For $\gamma = 2\hbar$ the formula (5.1) was obtained in [N] by means of scaling limit from the corresponding formula in XXZ model. In this particular case there exist a possibility to reduce the number of integrals in (5.1) by a method suggested by F.A. Smirnov [N]

We demonstrate this in the particular case of $m = 1$. Integral in (5.1) is interesting in this simplest case because it can be calculated explicitly for arbitrary step γ . We have in this particular case instead of (5.1) the following integral ($\alpha = \alpha_1 - \alpha_2$)

$$\begin{aligned} \frac{1}{\text{tr}(e^{\gamma d})} \text{tr}(e^{\gamma d} \Phi_{\nu_2}(\alpha_2) \Phi_{\nu_1}(\alpha_1)) &= \frac{(2\hbar)^{1/2}(-\hbar)}{\gamma^3 \Gamma^2\left(\frac{\gamma+\hbar}{\gamma}\right)} \frac{\Gamma\left(1 + \frac{\alpha_1 - \alpha_2}{2\hbar}\right)}{\Gamma\left(\frac{1}{2} + \frac{\alpha_1 - \alpha_2}{2\hbar}\right)} G(\alpha) \\ &\times \int_C \frac{du}{2\pi i} \frac{P_{\nu_2 \nu_1}(u)}{\gamma} \prod_{j=1}^2 \Gamma\left(\frac{u - \alpha_j}{\gamma}\right) \Gamma\left(\frac{\alpha_j + \hbar - u}{\gamma}\right) \end{aligned} \quad (5.2)$$

Denoting the integral in (5.2) by $I_{\nu_2 \nu_1}$ we can obtain that

$$I_{\nu_2 \nu_1} = -\nu_2 \gamma \frac{\Gamma\left(\frac{\gamma+\hbar}{\gamma}\right) \Gamma\left(\frac{\hbar}{\gamma}\right) \Gamma\left(\frac{\alpha+\hbar}{\gamma}\right) \Gamma\left(\frac{\gamma+\hbar-\alpha}{\gamma}\right)}{\Gamma\left(\frac{\gamma+2\hbar}{\gamma}\right)} \quad (5.3)$$

because of the following classical result (Mellin-Barnes type integral) [BE]

$$\int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i} \Gamma(a+s) \Gamma(b+s) \Gamma(c-s) \Gamma(d-s) = \frac{\Gamma(a+c) \Gamma(a+d) \Gamma(b+c) \Gamma(b+d)}{\Gamma(1+a+b+c+d)} \quad (5.4)$$

Hence

$$\frac{1}{\text{tr}(e^{\gamma d})} \text{tr}(e^{\gamma d} \Phi_{\nu_2}(\alpha_2) \Phi_{\nu_1}(\alpha_1)) = \frac{\nu_2 G(\alpha)}{(2\hbar)^{1/2}} \frac{\Gamma\left(1 + \frac{\alpha}{2\hbar}\right)}{\Gamma\left(\frac{3}{2} + \frac{\alpha}{2\hbar}\right)} \frac{\Gamma\left(\frac{\gamma+\hbar+\alpha}{\gamma}\right) \Gamma\left(\frac{\gamma+\hbar-\alpha}{\gamma}\right)}{\Gamma\left(\frac{\gamma+2\hbar}{\gamma}\right)}. \quad (5.5)$$

Applicability of the formula (5.4) to calculate the integral $I_{\nu_2 \nu_1}$ should be checked when $\gamma = 2\hbar$, because the integrand has a pole at the point ∞ in this case. Setting $\gamma = 2\hbar$ in (5.5) we obtain

$$\frac{1}{\text{tr}(e^{2\hbar d})} \text{tr}(e^{2\hbar d} \Phi_{\nu_2}(\alpha_2) \Phi_{\nu_1}(\alpha_1)) = \nu_2 (2\hbar)^{-1/2} G(\alpha) \Gamma\left(1 + \frac{\alpha}{2\hbar}\right) \Gamma\left(\frac{3}{2} - \frac{\alpha}{2\hbar}\right). \quad (5.6)$$

On the other hand we can calculate the integral $I_{\nu_2 \nu_1}|_{\gamma=2\hbar}$ using Smirnov's hint. Using the identity

$$\begin{aligned} \cos \pi \frac{\alpha_1 - \alpha_2}{2\hbar} &= \cos \pi \frac{u - \alpha_1}{2\hbar} \cos \pi \frac{u - \alpha_2}{2\hbar} + \sin \pi \frac{u - \alpha_1}{2\hbar} \sin \pi \frac{u - \alpha_2}{2\hbar} \\ &= \sin \pi \frac{u - \alpha_1 - \hbar}{2\hbar} \sin \pi \frac{u - \alpha_2 - \hbar}{2\hbar} + \sin \pi \frac{u - \alpha_1}{2\hbar} \sin \pi \frac{u - \alpha_2}{2\hbar} \end{aligned} \quad (5.7)$$

and the formula

$$\Gamma(x) \Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

we can reduce this integral as follows

$$I_{\nu_2 \nu_1} = \frac{\pi^2}{\cos \pi \frac{\alpha_1 - \alpha_2}{2\hbar}} \int_C \frac{du}{2\pi i} \frac{P_{\nu_2 \nu_1}(u)}{2\hbar} \prod_{j=1}^2 \frac{\Gamma\left(\frac{\hbar + \alpha_j - u}{2\hbar}\right)}{\Gamma\left(\frac{2\hbar + \alpha_j - u}{2\hbar}\right)} \quad (5.8)$$

$$+ \frac{\pi^2}{\cos \pi \frac{\alpha_1 - \alpha_2}{2\hbar}} \int_C \frac{du}{2\pi i} \frac{P_{\nu_2 \nu_1}(u)}{2\hbar} \prod_{j=1}^2 \frac{\Gamma\left(\frac{u - \alpha_j}{2\hbar}\right)}{\Gamma\left(\frac{\hbar + u - \alpha_j}{2\hbar}\right)} \quad (5.9)$$

Let us calculate first I_{+-} since arguments for the second integral I_{-+} are the same. Using explicit form of the polynomial $P_{+-}(u)$ we can find that the integral in question is equal to the sum of two integrals

$$I_{+-} = \frac{\pi^2}{\cos \pi \frac{\alpha_1 - \alpha_2}{2\hbar}} \int_{C_2} \frac{du}{2\pi i} \frac{\Gamma\left(\frac{1}{2} + \frac{\alpha_1 - u}{2\hbar}\right)}{\Gamma\left(\frac{\alpha_1 - u}{2\hbar}\right)} \frac{\Gamma\left(\frac{\hbar + \alpha_2 - u}{2\hbar}\right)}{\Gamma\left(\frac{1}{2} + \frac{\hbar + \alpha_2 - u}{2\hbar}\right)} \quad (5.10)$$

$$\begin{aligned}
& + \frac{\pi^2}{\cos \pi \frac{\alpha_1 - \alpha_2}{2\hbar}} \int_{C_1} \frac{du}{2\pi i} \frac{\Gamma\left(\frac{1}{2} + \frac{\hbar + u - \alpha_1}{2\hbar}\right)}{\Gamma\left(\frac{\hbar + u - \alpha_1}{2\hbar}\right)} \frac{\Gamma\left(\frac{u - \alpha_2}{2\hbar}\right)}{\Gamma\left(\frac{1}{2} + \frac{u - \alpha_2}{2\hbar}\right)} \\
& = -\frac{\pi^2}{\cos \pi \frac{\alpha}{2\hbar}} \frac{\alpha - \hbar}{2} = \pi \hbar \Gamma\left(\frac{1}{2} + \frac{\alpha}{2\hbar}\right) \Gamma\left(\frac{3}{2} - \frac{\alpha}{2\hbar}\right)
\end{aligned} \tag{5.11}$$

where contours C_1 and C_2 being small semicircles around the point ∞ are shown on the Fig. 1.



Fig. 1.

Since the arguments in ratios of Γ -functions (5.10), (5.11) differ by $1/2$ we can calculate these integrals using the Stirling formula. The same calculation for the integral I_{-+} shows that formula (5.5) is correct for all possible values of γ .

Also one can easily check that (5.5) is in accordance with the normalization conditions for the type I intertwining operators (3.17) and (3.18) which in this case looks as

$$\Phi_{\nu_2}(\alpha) \Phi_{\nu_1}(\alpha - \hbar) = \nu_2 g^{-1} \delta_{\nu_2, -\nu_1}$$

and

$$\sum_{\nu} \nu \Phi_{\nu}(\alpha - \hbar) \Phi_{-\nu}(\alpha) = g^{-1}.$$

6 Calculation of Multi-Point Traces for Type II VO

The goal of this section is to prove the statement that in the “physical” limit $\gamma = 2\hbar$ the trace of the product of type II intertwining operators is equal identically zero. In such a case we have from (4.11) for $M = m = 0$ and $N = 2n$ due to (4.10):

$$\begin{aligned}
& \frac{1}{\text{tr}(e^{2\hbar d})} \text{tr} \left(e^{2\hbar d} \prod_{j=1}^{2n} \Psi_{\varepsilon_i}^*(\beta_i) \right) = A_{2n,0}^{n,0} \prod_{j < j'} \tilde{\zeta}(\beta_j - \beta_{j'}) \\
& \times \prod_{k=1}^n \int_{\tilde{C}} \frac{dv_k}{2\pi i} \prod_{j=1}^{2n} \Gamma\left(\frac{\beta_j - v_k}{2\hbar}\right) \Gamma\left(\frac{v_k - \beta_j - \hbar}{2\hbar}\right) \frac{\tilde{P}_{\varepsilon}(v; \beta)}{(2\hbar)^{n(2n-1)}} \\
& \times \prod_{k < k'} \frac{\sin \pi \frac{v_k - v_{k'}}{2\hbar}}{\Gamma\left(\frac{v_k - v_{k'} - \hbar}{2\hbar}\right) \Gamma\left(\frac{v_{k'} - v_k + \hbar}{2\hbar}\right)}
\end{aligned} \tag{6.1}$$

Using the identity

$$\begin{aligned}
& \prod_{i < j} \frac{\sin \pi \frac{v_i - v_j}{2\hbar}}{\Gamma\left(\frac{v_i - v_j - \hbar}{2\hbar}\right) \Gamma\left(\frac{v_j - v_i + \hbar}{2\hbar}\right)} = \left(\frac{\pi i}{4\hbar}\right)^{n(n-1)/2} \frac{2^{2n-1} \pi^{2n}}{\tilde{B}_1 B_{2n}^{1/2}} \\
& \times \sum_{k=1}^n \Delta_k(v_j; j \neq k) \prod_{\substack{i < j \\ i, j \neq k}} (v_i - v_j - \hbar) \prod_{j=1}^{k-1} (v_k - v_j + \hbar) \prod_{j=k+1}^n (v_k - v_j - \hbar) \\
& \times \left[\frac{1}{\prod_{j=1}^{2n} \Gamma\left(\frac{v_k - \beta_j - \hbar}{2\hbar}\right) \Gamma\left(\frac{\beta_j - v_k + 3\hbar}{2\hbar}\right)} - \frac{(-1)^n}{\prod_{j=1}^{2n} \Gamma\left(\frac{v_k - \beta_j}{2\hbar}\right) \Gamma\left(\frac{\beta_j - v_k + 2\hbar}{2\hbar}\right)} \right]
\end{aligned}$$

where

$$\tilde{B}_1 = \sum_j e^{-\pi i \beta_j / \hbar}, \quad B_{2n} = \exp \left(\frac{\pi i}{\hbar} \sum_j \beta_j \right)$$

$$\Delta_k(v_j; j \neq k) = \det \left(e^{-(n-2l+1)\pi i v_j / \hbar} \right)_{2 \leq l \leq n, 1 \leq j \neq k \leq n}$$

we can rewrite the integral in (6.1) over the variable v_k as a difference of two integrals

$$\int_{\tilde{C}} \frac{dv_k}{2\pi i} f(v_k) \left[\prod_{j=1}^{2n} \frac{\Gamma \left(\frac{\beta_j - v_k}{2\hbar} \right)}{\Gamma \left(\frac{3\hbar + \beta_j - v_k}{2\hbar} \right)} - (-1)^n \prod_{j=1}^{2n} \frac{\Gamma \left(\frac{v_k - \beta_j - \hbar}{2\hbar} \right)}{\Gamma \left(\frac{v_k - \beta_j + 2\hbar}{2\hbar} \right)} \right] \quad (6.2)$$

where we have introduced a polynomial $f(v_k)$

$$f(v) = \prod_{j=1}^{k-1} (v_k - \beta_j - \hbar) \prod_{j=k+1}^{2n} (v_k - \beta_j) \prod_{j=1}^{k-1} (v_k - v_j + \hbar) \prod_{j=k+1}^n (v_k - v_j - \hbar)$$

We can see now that the integral in (6.2) is equal to the sum of two integrals over two small semicircles around the infinity point. These integrals are reduced to residues of the integrand at this point. Using the Stirling formula one can easily notice that the expansion of the integrand in (6.2) in the vicinity of the point ∞ starts from v_k^{-2} . Indeed, the total power of polynomial $f(v_k)$ is equal $3n - 2$, while the ratio of Γ -functions produce v_k^{3n} in the denominator. Since it is true for all terms in the sum over k we proved that the trace

$$\frac{1}{\text{tr}(e^{2\hbar d})} \text{tr} \left(e^{2\hbar d} \prod_{j=1}^{2n} \Psi_{\varepsilon_i}^*(\beta_j) \right) = 0 \quad (6.3)$$

It is instructive to consider again the simplest case of the formula (6.1) for $n = 1$. This integral can be calculated using (5.4) and is equal to $(\beta = \beta_1 - \beta_2)$

$$\frac{1}{\text{tr}(e^{2\hbar d})} \text{tr} (e^{\gamma d} \Psi_{\varepsilon_2}^*(\beta_2) \Psi_{\varepsilon_1}^*(\beta_1)) = \varepsilon_1 (2\hbar)^{1/2} \tilde{G}(\beta) \frac{\Gamma \left(\frac{1}{2} + \frac{\beta}{2\hbar} \right) \Gamma \left(\frac{\beta - \hbar}{\gamma} \right) \Gamma \left(\frac{\gamma - \beta - \hbar}{\gamma} \right)}{\Gamma \left(\frac{\beta}{2\hbar} \right) \gamma \Gamma \left(\frac{\gamma - 2\hbar}{\gamma} \right)} \quad (6.4)$$

We see that the vanishing of this trace for $\gamma = 2\hbar$ is due to the function $\Gamma \left(1 - \frac{2\hbar}{\gamma} \right)$ in denominator of (6.4).

We can check that (6.4) is in accordance with the normalization condition for the type II intertwining operators

$$\Psi_{\varepsilon_2}^*(\beta_2) \Psi_{\varepsilon_1}^*(\beta_1) = \frac{\varepsilon_1 g \delta_{\varepsilon_1, -\varepsilon_2}}{\beta_1 - \beta_2 - \hbar} + o(\beta_1 - \beta_2 - \hbar) \quad (6.5)$$

One can see that the function (6.4) indeed has a pole at the point $\beta_1 = \beta_2 + \hbar$ with the residue equal to $\varepsilon_1 g$.

7 Deformation of Gauss-Manin Connection

As we have seen in the previous sections the general formula for the trace simplifies much in the case when $\gamma = 2\hbar$. In this case the trace of the product of type II intertwining operators appear to be equals zero identically. To make a trace of type II vertex operators non-zero for $\gamma = 2\hbar$ we should add some combination of type I operators. In this and the next sections we are going to consider only simplest cases of the formula (4.11), while general situation will be considered elsewhere. Our goal is to identify some particular cases of the general trace formula (4.11) with those obtained by bootstrap approach for $SU(2)$ -invariant Thirring model. On the way of the identification we will find some identities between

form factor integrals which turns out to be deformations of the Gauss-Manin connection identities for the hyperelliptic curves.

Let us start with the case of one type I intertwining operator in the formula (4.11) (which does not correspond to local operators). For simplicity we consider this operator to be $\Phi_-(\alpha)$ component of the intertwining operator. This means that $M = 1$ and $m = 0$ in (4.11). The number N and n which describe the total number of type II operators and “plus” components of them respectively are related because of the condition (4.10)

$$N = 2n + 1. \quad (7.1)$$

The case $N = 1$ is trivial. The general formula (4.11) does not contain integration at all. First nontrivial case is $N = 3$ and $n = 1$. In such a case the formula (4.11) reads

$$\begin{aligned} \frac{1}{\text{tr}(e^{2\hbar d})} \text{tr} \left(e^{2\hbar d} \prod_{j=1}^3 \Psi_{\varepsilon_j}^*(\beta_j) \Phi_-(\alpha) \right) &= 2\hbar A_{3,1}^{1,0} \prod_{j < j'} \tilde{\zeta}(\beta_j - \beta_{j'}) \prod_j \bar{\zeta}(\alpha - \beta_j) \\ &\times \int_{\tilde{C}} \frac{dv}{2\pi i} \prod_{j=1}^3 \Gamma\left(\frac{\beta_j - v}{2\hbar}\right) \Gamma\left(\frac{v - \beta_j + \hbar}{2\hbar}\right) \frac{\tilde{P}_\varepsilon(v; \beta)}{\prod_{j=1}^3 (v - \beta_j - \hbar)} \sin \pi \frac{v - \alpha}{2\hbar} \end{aligned} \quad (7.2)$$

The contour \tilde{C} in the integral (7.2) is shown on the Fig. 2.

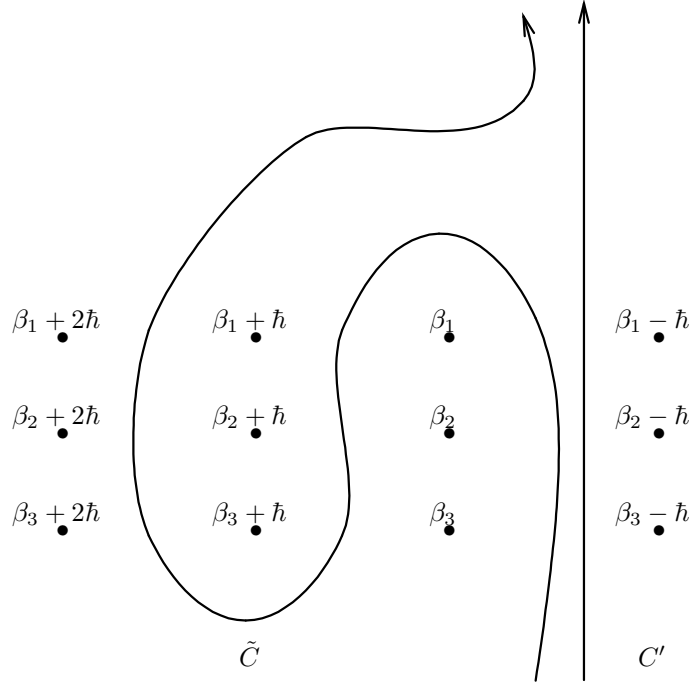


Fig. 2.

The form of the contour \tilde{C} and presence of “strange” poles (from bootstrap approach point of view) at the points $\beta_j + \hbar$ in the integral (7.2) were main obstacles in identification of free field [JM] and bootstrap [S1] formulas for form factors. We would like to state that both these obstacles can be overcome because of the following

Proposition 3. *For $m = 1, 2, 3$ we have identities*

$$\begin{aligned} \prod_{j \neq m} (\beta_m + \hbar - \beta_j) \int_{\tilde{C}} \frac{dv}{2\pi i} \prod_{j=1}^3 \varphi(v - \beta_j) \frac{1}{v - \beta_m - \hbar} \exp\left(\pm i \frac{\pi v}{2\hbar}\right) \\ = \int_{C'} \frac{dv}{2\pi i} \left[\prod_{j=1}^3 \varphi(v - \beta_j) \right] (v - \beta_m) \exp\left(\pm i \frac{\pi v}{2\hbar}\right) \end{aligned} \quad (7.3)$$

where

$$\varphi(v) = \Gamma\left(-\frac{v}{2\hbar}\right) \Gamma\left(\frac{1}{2} + \frac{v}{2\hbar}\right) \quad (7.4)$$

which satisfy

$$\varphi(v + 2\hbar) = -\frac{v + \hbar}{v + 2\hbar} \varphi(v) \quad (7.5)$$

and the contour C' is a straight line between the points β_j and $\beta_j - \hbar$, $j = 1, 2, 3$ and shown on the Fig. 2.

The identity formulated in (7.3) is of a type of the total difference identities found by F.A. Smirnov in [S3, S4]. The proof of (7.3) is based on the property (7.5) of the function $\varphi(v)$.

Hence due to (7.3) the extremal component $(\varepsilon_3, \varepsilon_2, \varepsilon_1 = +--)$ of the three-particle form factor of the operator $\Phi_-(\alpha)$ is equal to

$$\begin{aligned} & \frac{1}{\text{tr}(e^{2\hbar d})} \text{tr}(e^{2\hbar d} \Psi_+^*(\beta_3) \Psi_-^*(\beta_2) \Psi_-^*(\beta_1) \Phi_-(\alpha)) \\ &= 4\hbar A_{3,1}^{1,0} \prod_{j < j'} \tilde{\zeta}(\beta_j - \beta_{j'}) \prod_j \bar{\zeta}(\alpha - \beta_j) \prod_{j=1}^2 \frac{1}{(\beta_3 + \hbar - \beta_j)} \\ & \times \exp\left(\frac{i\pi}{4\hbar} \left(3\alpha + \sum_{j=1}^3 \beta_j\right)\right) \left[\prod_{j=1}^3 \sin \pi \frac{\alpha - \beta_j}{4\hbar} + i \prod_{j=1}^3 \cos \pi \frac{\alpha - \beta_j}{4\hbar} \right] \\ & \times \int_{C'} \frac{dv}{2\pi i} \left[\prod_{j=1}^3 \varphi(v - \beta_j) \right] (v - \beta_3) \exp\left(-\frac{i\pi v}{2\hbar}\right) \end{aligned} \quad (7.6)$$

The formula (7.6) coincides with corresponding formula from [JKMQ]. To obtain (7.6) an obvious identity

$$\int_{C'} \frac{dv}{2\pi i} \left[\prod_{j=1}^3 \varphi(v - \beta_j) \right] (v - \beta_3) \left[\exp\left(\frac{i\pi v}{2\hbar}\right) + B_2 \exp\left(-\frac{i\pi v}{2\hbar}\right) \right] = 0 \quad (7.7)$$

have been used, where

$$B_2 = \exp\left(\sum_{j=1}^3 \frac{i\pi\beta_j}{2\hbar}\right) \sum_{j=1}^3 \exp\left(-\frac{i\pi\beta_j}{2\hbar}\right).$$

Another important application of the identities (7.3) is that they provide a deformation of Gauss-Manin connection for elliptic curves [M]. To explain what we mean let us define two integrals

$$F_p^{(k)}(\beta_1, \beta_2, \beta_3) = \int_{C'} \frac{v^p dv}{2\pi i} \prod_{j=1}^3 \varphi(v - \beta_j) \exp\left(ik \frac{\pi v}{2\hbar}\right), \quad p = 0, 1 \quad k = \pm \quad (7.8)$$

which are deformed analogs of first and second kind integrals on elliptic curve [S3]. Due to (7.3) the integrals (7.8) satisfy the difference equation

$$F_p^{(k)}(\beta_1, \beta_2, \beta_3 + 2\hbar) + \sum_{r=1}^2 A_{pr}(\beta_1, \beta_2, \beta_3) F_r^{(k)}(\beta_1, \beta_2, \beta_3) = 0 \quad (7.9)$$

with the connection matrix

$$A(\beta) = \begin{pmatrix} 1 & 0 \\ \hbar & 1 \end{pmatrix} + \frac{1}{\prod_{j=1}^2 (\beta_3 + \hbar - \beta_j)} \begin{pmatrix} \hbar\beta_3 & \hbar \\ \hbar\beta_3(\beta_3 + \hbar) & \hbar(\beta_3 + \hbar) \end{pmatrix} \quad (7.10)$$

Difference equation (7.9) is a deformed analog of the Gauss-Manin connection. Classical limit ($\hbar \rightarrow 0$) of the integrals $F_p^{(k)}(\beta_1, \beta_2, \beta_3)$ is not straightforward and was investigated in [S3, S4]. In such limit these integrals go to elliptic integrals

$$F_p^{(+)}(\beta_1, \beta_2, \beta_3) \underset{\hbar \rightarrow 0}{\sim} \int_{\beta_1}^{\beta_2} \frac{v^p dv}{\sqrt{\prod_{j=1}^3 (v - \beta_j)}}$$

$$F_p^{(-)}(\beta_1, \beta_2, \beta_3) \underset{\hbar \rightarrow 0}{\sim} \int_{\beta_2}^{\beta_3} \frac{v^p dv}{\sqrt{\prod_{j=1}^3 (v - \beta_j)}}$$

and difference equation (7.9) to the classical Gauss-Manin connection [M]. Identity (7.3) becomes the identity between elliptic integrals of the type

$$\prod_{j \neq m} (\beta_3 - \beta_j) \int_{\beta_1}^{\beta_2} \frac{dv}{\sqrt{\prod_{j=1}^3 (v - \beta_j)(v - \beta_3)}} = \int_{\beta_1}^{\beta_2} \frac{(v - \beta_3)dv}{\sqrt{\prod_{j=1}^3 (v - \beta_j)}} \quad (7.11)$$

which describe a relation between second kind elliptic integrals with singularities at the point β_3 and at the point ∞ .

The next example which is interesting from physical point of view is two type I intertwining operators in the trace formula (4.11). The case when $m = n = 0$ and $M = N = 2$ is trivial because in this case (4.11) does not contain integrations. We can state that the integral (4.11) can be reduced to form factor integrals in $SU(2)$ -invariant Thirring model for $N = 2n$, $M = 2m = 2$, $\gamma = 2\hbar$ and arguments of type I operators are related as

$$\alpha_1 = \alpha_2 + \hbar. \quad (7.12)$$

Because of the property (3.17) this case correspond to $2n$ -particle form factor of the generating function of the local operators [L]

$$\Lambda(\alpha) = \Phi_+(\alpha)\Phi_-(\alpha + \hbar) + \Phi_-(\alpha)\Phi_+(\alpha + \hbar) = \Phi_-(\alpha)\Phi_-(\alpha + \hbar)f_0 - f_0\Phi_-(\alpha)\Phi_-(\alpha + \hbar) \quad (7.13)$$

The integral formula (4.11) reduces in this case to the multiple integral

$$\begin{aligned} & \frac{1}{\text{tr}(e^{2\hbar d})} \text{tr} \left(e^{2\hbar d} \prod_{j=1}^{2n} \Psi_{\varepsilon_j}^*(\beta_j) \Phi_+(\alpha) \Phi_-(\alpha + \hbar) \right) \\ &= \frac{A_{2n,2}^{n,1} \pi^{3/2} (-1)^n}{2^{n+1} (2\hbar)^{n(n-3)/2} G^{1/2}(0)} \frac{\prod_{j < j'} \tilde{\zeta}(\beta_j - \beta_{j'})}{\prod_{j=1}^{2n} \cos \pi \frac{\alpha - \beta_j}{2\hbar}} \\ & \prod_{k=1}^n \int_{\tilde{C}} \frac{dv_k}{2\pi i} \prod_{j=1}^{2n} \frac{\varphi(v_k - \beta_j)}{(v_k - \beta_j - \hbar)} \tilde{P}_\varepsilon(v; \beta) \prod_{k < k'} (v_k - v_{k'} - \hbar) \\ & \times \prod_{k < k'} \sin \pi \frac{v_k - v_{k'}}{\hbar} \prod_{k=1}^n \sin \pi \frac{v_k - \alpha}{\hbar} \int_C \frac{du}{2\pi i} \frac{\prod_{j=1}^{2n} \sin \pi \frac{u - \beta_j}{2\hbar}}{\prod_{k=1}^n \sin \pi \frac{u - v_k}{\hbar} \sin \pi \frac{u - \alpha}{\hbar}} \end{aligned} \quad (7.14)$$

Note that in order to set $\alpha_1 = \alpha_2 + \hbar$ in the general formula (4.11) we have used the explicit form of the polynomial $P_{+-}(u)$ which cancel the pole of the function $\Gamma\left(\frac{u - \alpha_1}{2\hbar}\right)$ at the point α_1 . It means that there is no pinching of the contour C between points $\alpha_2 + \hbar$ and α_1 when $\alpha_1 \rightarrow \alpha_2 + \hbar$. The poles of the integrand over u at the points

$$\alpha + r\hbar, \quad v + r\hbar, \quad k = 1, \dots, n$$

are inside the contour C for $r = 1, 2, \dots$ and are outside C for $r = 0, -1, -2, \dots$. Note also that all Γ -function depending on the integration variable u turns into sin-functions, so we can calculate the integral over u . After this we can see that one more integration over variables v_k can be calculated and the result of calculation coincides with form factors integral obtained in [KS, S1] for $SU(2)$ -invariant Thirring model.

Unfortunately, this treatment of the integral (7.14) is technically complicated problem. We are going to publish this calculation as a separate paper [KLP2]. Here we would like to point out the main steps of our construction.

First step is the calculation of the integral over u in (7.14) which can be done easily. Second one is the calculation of the integral over one of variables v_k using Smirnov's hint (see section 6). After these calculations we left with $n - 1$ integrals over the contours \tilde{C} of the type shown on Fig. 2. In order to rectify contours we should use the identities which are analogs of the Gauss-Manin connection identities. In the simplest case $n = 2$ this identity reads

$$\begin{aligned} & \int_{\tilde{C}} \frac{dv}{2\pi i} \frac{\prod_{j=1}^4 \varphi(v - \beta_j)}{v - \beta_4 - \hbar} \exp \pi \left(\pm \frac{i\pi v}{\hbar} \right) \\ &= \frac{1}{\prod_{j \neq 4} (\beta_4 - \beta_j + \hbar)} \int_{C'} \frac{dv}{2\pi i} d(v, \beta) \prod_{j=1}^4 \varphi(v - \beta_j) \exp \left(\pm \frac{i\pi v}{\hbar} \right) \\ & d(v, \beta) = 2v^2 - v \left(2\hbar - \sum_{j=1}^4 \beta_j \right) + \beta_4(\beta_1 + \beta_2 + \beta_3 - \beta_4 - 2\hbar) \end{aligned} \quad (7.15)$$

The contours \tilde{C} and C' in (7.15) are the same as shown on the Fig. 2. These identities can be easily generalized to arbitrary $n \geq 3$ and are the analogs of the Gauss-Manin connection identities for the hyperelliptic curves of the genus $n - 1$. The last step of identification (7.14) with form factor integrals is using the identities of the type (7.7). The identities of this kind allow one to extract from (7.14) the integrals which do not depend on the spectral parameter α of the generating function $\Lambda(\alpha)$. Realization of such a program for the trace

$$\frac{\text{tr} \left(e^{2\hbar d} \Psi_+^*(\beta_4) \Psi_+^*(\beta_3) \Psi_-^*(\beta_2) \Psi_-^*(\beta_1) \Lambda(\alpha) \right)}{\text{tr} \left(e^{2\hbar d} \right)}$$

leads to the integral

$$\begin{aligned} & \prod_{j < j'} \tilde{\zeta}(\beta_j - \beta_{j'}) \prod_{\substack{i=1,2 \\ j=3,4}} \frac{1}{(\beta_j - \beta_i + \hbar)} \\ & \times \left(\sum_{j=1}^4 \exp \frac{i\pi \beta_j}{\hbar} \right)^{-1} \exp \left(\sum_{j=1}^4 \frac{i\pi \beta_j}{2\hbar} \right) \\ & \times \int_{C'} \frac{dv}{2\pi i} \prod_{j=1}^4 \varphi(\beta_j - v) \exp \left(-\frac{i\pi v}{\hbar} \right) \left[\prod_{j=1,2} (v - \beta_j + \hbar) + \prod_{j=3,4} (v - \beta_j) \right] \end{aligned} \quad (7.16)$$

Formula (7.16) coincide with analogous formula from [S2].

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